Topic 9 - Poisson Distribution

Statistics for Managers

June 3, 1999

The Poisson distribution describes the probability of the number of times that a random event will occur in a time or space interval under the conditions that

- The probability of the event occurring is very small,
- The number of trials is very large so that the event actually occurs a few times.

The expected number of events occurring in the time interval is μ.

Examples

- The number of defects in high-tension cable used in suspension bridge construction. A good manufacturing process achieves an expected number of defects , μ, equal to only two defects per 100 meters.
- The number of children having leukemia in a large metropolitan region. Under normal circumstances, childhood leukemia occurs at the expected number μ equal to one in 10,000 children.
- The number of undercharge errors in one month's accounts of a large industrial organization. In poor accounting systems, the expected number of undercharges could be as high as μ equal to one in five accounts.
- The number of auto insurance claims of a certain type in one week at a large insurance company. Under normal circumstances, the expected number of car thefts is μ equal to five per week.

Formula for the Poisson Probability Mass Function

The number of events Y occurring in one unit of time or space is Poisson (μ):

$$f(y) = \frac{\mu^{\mu} e^{-\mu}}{y!}, y = 0,1,2,3,\ldots$$

The mean and variance of the four samples is approximately μ.

Mean and variance:

$$E(Y) = \mu$$
$$VAR(Y) = \mu$$

Example 1. Cable manufacturing

In cable manufacturing with μ = 2 defects per 100 meters, the probability of Y = 1 defect in 100 meters of cable is

$$f(1) = \frac{2^1 e^{-2}}{1!} = 2 \times 0.367879 \cong 0.74$$

Example 2. Leukemia cases

In children with the mean μ = 1 leukemia case in 10,000, the probability of Y = 10 cases is

$$f(10) = \frac{1^{10} e^{-1}}{10!} \cong \frac{1 \times 0.367879441}{3,628,800} \cong 0.000\,000\,101\,4$$

The probability of Y = 0 cases is

$$f(0) = \frac{1^0 e^{-1}}{0!} \cong \frac{1 \times 0.367879441}{1} \cong 0.3679$$

The probability of at least one case is

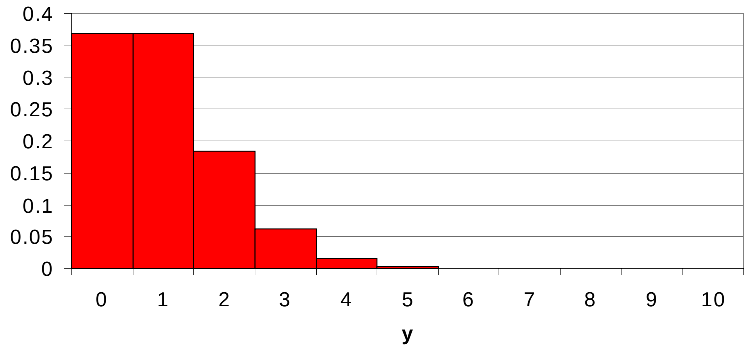
$$1 - f(0) \cong 1 - 0.3679 = 0.6321$$

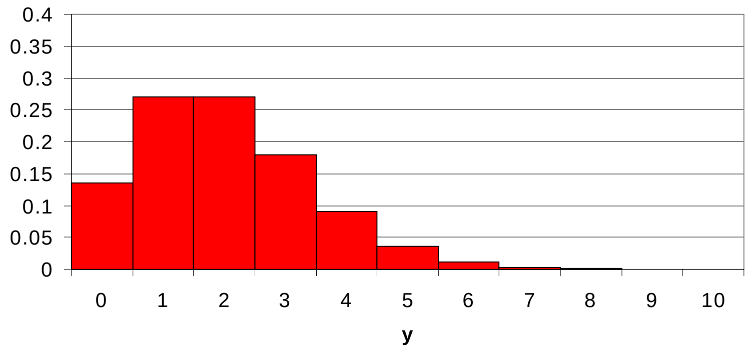
Example 3. Undercharge

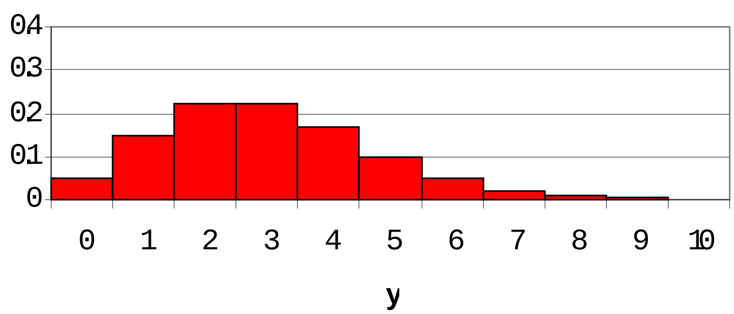
The mean number of billing undercharges is one per five accounts. Therefore, in 20 accounts, the applicable mean is μ equal to four in 20 accounts. The probability of no undercharges is

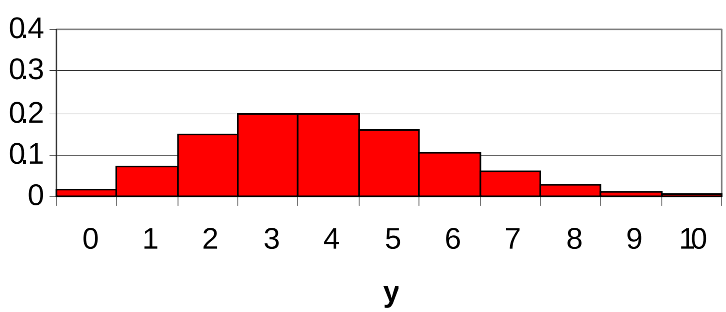
$$f(0) = \frac{4^0 e^{-4}}{0!} = \frac{1 \times e^{-4}}{1} \cong 0.01831564$$

Histograms of the Poisson distribution with μ = 1, 2, 3, 4









Probability tables of the Poisson distribution with μ = 1, 2, 3, 4

	μ			
	1	2	3	4
y	f(y)	f(y)	f(y)	f(y)
0	0.367879	0.135335	0.049787	0.018316
1	0.367879	0.270671	0.149361	0.073263
2	0.183940	0.270671	0.224042	0.146525
3	0.061313	0.180447	0.224042	0.195367
4	0.015328	0.090224	0.168031	0.195367
5	0.003066	0.036089	0.100819	0.156293
6	0.000511	0.012030	0.050409	0.104196
7	7.3E-05	0.003437	0.021604	0.059540
8	9.12E-06	0.000859	0.008102	0.029770
9	1.01E-06	0.000191	0.002701	0.013231
10	1.01E-07	3.82E-05	0.000810	0.005292
11	9.22E-09	6.94E-06	0.000221	0.001925
12	7.68E-10	1.16E-06	5.52E-05	0.000642
13	5.91E-11	1.78E-07	1.27E-05	0.000197
14	4.22E-12	2.54E-08	2.73E-06	5.64E-05
ETC.				

Excel formulas for Poisson probability mass function with μ = 2

	μ = 2
y	f(y)
0	= POISSON(0,2,0)
1	= POISSON(1,2,0)
2	= POISSON(2,2,0)
3	= POISSON(3,2,0)
4	= POISSON(4,2,0)
5	= POISSON(5,2,0)
6	= POISSON(6,2,0)
7	= POISSON(7,2,0)
8	= POISSON(8,2,0)

Example 4. Cumulative distribution function and the undercharge example

Sometimes, it is desired to find the probability that the number of events is less than or equal to a given number. Obviously, the answer involves summing the individual probabilities. For example, the probability of 3 or less undercharges is F(3) = f(0) + f(1) + f(2) + f(3). Computationally, it is easier to use the cumulative distribution function directly rather than summing several terms of the probability mass function. The probability of three or less under charges is F(3) = 0.433 found in Table 1. Poisson cumulative distribution function. Therefore the probability of four or more undercharges is 1 – 0.433 = 0.567

It is possible to use the cumulative distribution function to evaluate the probability mass function itself. For example, the probability of exactly three undercharges is f(3) = F(3) – F(2) = 0.433 – 0.238 = 0.195.

Poisson Approximation to the Binomial Distribution

If n is large and $np < 10$, then BIN (n, p) is approximately Poisson (np). Thus the number of successes Y in n Bernoulli trials is approximately

$$f(y) = \frac{(np)^y e^{-np}}{y!}, y = 0,1,2,3,\ldots$$

Example 5. Auto parts

The probability of a defective part is 0.1 percent, in a shipment of 1000 auto parts, what is the probability that there is one defective part? The BIN (1000, 0.001) is approximately Poisson with $\lambda = np = (1000)(.001) = 1$.

Thus, the probability that there is one defective part is approximately

$$f(1) = \frac{1^1 e^{-1}}{1!} \cong \frac{1 \times 0.367879441}{1} \cong 0.3679$$

The probability that there are no defective parts is approximately

$$f(0) = \frac{0^1 e^{-1}}{0!} \cong \frac{1 \times 0.367879441}{1} \cong 0.3679$$

The probability that there is at least one defective part is approximately

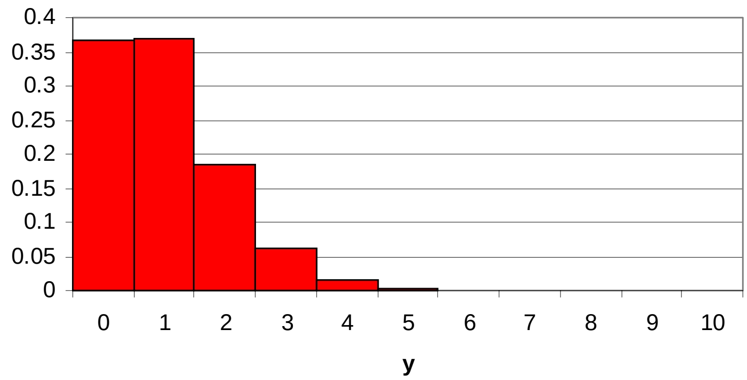
$$1 - f(0) \cong 1 - 0.3679 = 0.6321$$

Example 6. Auto parts continued

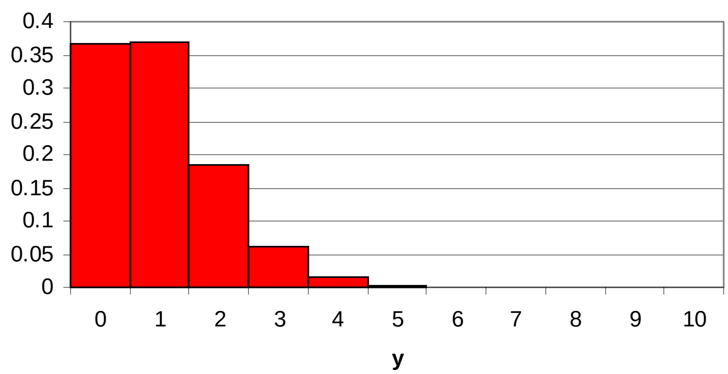
The probability that there are at least two but no more than five defective parts is f(2) + f(3) + f(4) + f(5) = F(5) – F(1) = 0.999 – 0.736 = 0.263. We have used Table 1. Poisson cumulative distribution function.

Example 7. Binomial distributions for increasing n and decreasing p

Histogram of BIN (100, 0.01)



Histogram of BIN (500, 0.002)



Histogram of BIN (1000, 0.001)

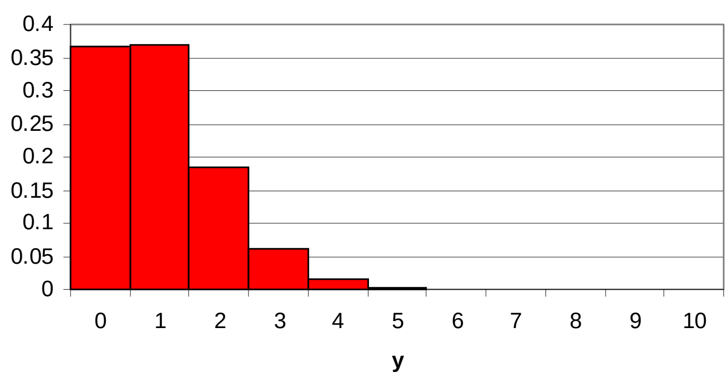
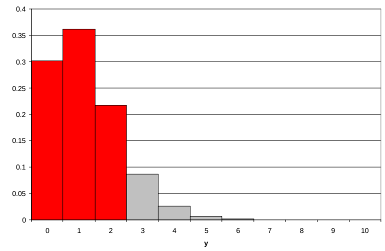


Table 1. Poisson Cumulative Distribution Function

Entries are $F(y) = P(Y \le y)$

Example:

To find F(3), $\mu = 1.2$, look up cell with row “1.2” and column “3,” which is 0.879.



μ\y	0	1	2	3	4	5	6	7	8	9	10
0.05	0.951	0.999	1.000								
0.10	0.905	0.995	1.000								
0.15	0.861	0.990	0.999	1.000							
0.20	0.819	0.982	0.999	1.000							
0.25	0.779	0.974	0.998	1.000							
0.30	0.741	0.963	0.996	1.000							
0.35	0.705	0.951	0.994	1.000							
0.40	0.670	0.938	0.992	0.999	1.000						
0.45	0.638	0.925	0.989	0.999	1.000						
0.50	0.607	0.910	0.986	0.998	1.000						
0.55	0.577	0.894	0.982	0.998	1.000						
0.60	0.549	0.878	0.977	0.997	1.000						
0.65	0.522	0.861	0.972	0.996	0.999	1.000					
0.70	0.497	0.844	0.966	0.994	0.999	1.000					
0.75	0.472	0.827	0.959	0.993	0.999	1.000					
0.80	0.449	0.809	0.953	0.991	0.999	1.000					
0.85	0.427	0.791	0.945	0.989	0.998	1.000					
0.90	0.407	0.772	0.937	0.987	0.998	1.000					
0.95	0.387	0.754	0.929	0.984	0.997	1.000					
1.0	0.368	0.736	0.920	0.981	0.996	0.999	1.000				
1.1	0.333	0.699	0.900	0.974	0.995	0.999	1.000				
1.2	0.301	0.663	0.879	0.966	0.992	0.998	1.000				
1.3	0.273	0.627	0.857	0.957	0.989	0.998	1.000				
1.4	0.247	0.592	0.833	0.946	0.986	0.997	0.999	1.000			
1.5	0.223	0.558	0.809	0.934	0.981	0.996	0.999	1.000			
1.6	0.202	0.525	0.783	0.921	0.976	0.994	0.999	1.000			
1.7	0.183	0.493	0.757	0.907	0.970	0.992	0.998	1.000			
1.8	0.165	0.463	0.731	0.891	0.964	0.990	0.997	0.999	1.000		
1.9	0.150	0.434	0.704	0.875	0.956	0.987	0.997	0.999	1.000		
2.0	0.135	0.406	0.677	0.857	0.947	0.983	0.995	0.999	1.000		
2.2	0.111	0.355	0.623	0.819	0.928	0.975	0.993	0.998	1.000		
2.4	0.091	0.308	0.570	0.779	0.904	0.964	0.988	0.997	0.999	1.000	
2.6	0.074	0.267	0.518	0.736	0.877	0.951	0.983	0.995	0.999	1.000	
2.8	0.061	0.231	0.469	0.692	0.848	0.935	0.976	0.992	0.998	0.999	1.000

Table 1. Poisson Cumulative Distribution Function Continued

µ\y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3.0	0.050	0.199	0.423	0.647	0.815	0.916	0.966	0.988	0.996	0.999	1.000										
3.2	0.041	0.171	0.380	0.603	0.781	0.895	0.955	0.983	0.994	0.998	1.000										
3.4	0.033	0.147	0.340	0.558	0.744	0.871	0.942	0.977	0.992	0.997	0.999	1.000									
3.6	0.027	0.126	0.303	0.515	0.706	0.844	0.927	0.969	0.988	0.996	0.999	1.000									
3.8	0.022	0.107	0.269	0.473	0.668	0.816	0.909	0.960	0.984	0.994	0.998	0.999	1.000								
4.0	0.018	0.092	0.238	0.433	0.629	0.785	0.889	0.949	0.979	0.992	0.997	0.999	1.000								
4.2	0.015	0.078	0.210	0.395	0.590	0.753	0.867	0.936	0.972	0.989	0.996	0.999	1.000								
4.4	0.012	0.066	0.185	0.359	0.551	0.720	0.844	0.921	0.964	0.985	0.994	0.998	0.999	1.000							
4.6	0.010	0.056	0.163	0.326	0.513	0.686	0.818	0.905	0.955	0.980	0.992	0.997	0.999	1.000							
4.8	0.008	0.048	0.143	0.294	0.476	0.651	0.791	0.887	0.944	0.975	0.990	0.996	0.999	1.000							
5.0	0.007	0.040	0.125	0.265	0.440	0.616	0.762	0.867	0.932	0.968	0.986	0.995	0.998	0.999	1.000						
5.2	0.006	0.034	0.109	0.238	0.406	0.581	0.732	0.845	0.918	0.960	0.982	0.993	0.997	0.999	1.000						
5.4	0.005	0.029	0.095	0.213	0.373	0.546	0.702	0.822	0.903	0.951	0.977	0.990	0.996	0.999	1.000						
5.6	0.004	0.024	0.082	0.191	0.342	0.512	0.670	0.797	0.886	0.941	0.972	0.988	0.995	0.998	0.999	1.000					
5.8	0.003	0.021	0.072	0.170	0.313	0.478	0.638	0.771	0.867	0.929	0.965	0.984	0.993	0.997	0.999	1.000					
6.0	0.002	0.017	0.062	0.151	0.285	0.446	0.606	0.744	0.847	0.916	0.957	0.980	0.991	0.996	0.999	0.999	1.000				
6.2	0.002	0.015	0.054	0.134	0.259	0.414	0.574	0.716	0.826	0.902	0.949	0.975	0.989	0.995	0.998	0.999	1.000				
6.4	0.002	0.012	0.046	0.119	0.235	0.384	0.542	0.687	0.803	0.886	0.939	0.969	0.986	0.994	0.997	0.999	1.000				
6.6	0.001	0.010	0.040	0.105	0.213	0.355	0.511	0.658	0.780	0.869	0.927	0.963	0.982	0.992	0.997	0.999	0.999	1.000			
6.8	0.001	0.009	0.034	0.093	0.192	0.327	0.480	0.628	0.755	0.850	0.915	0.955	0.978	0.990	0.996	0.998	0.999	1.000			
7.0	0.001	0.007	0.030	0.082	0.173	0.301	0.450	0.599	0.729	0.830	0.901	0.947	0.973	0.987	0.994	0.998	0.999	1.000			
7.2	0.001	0.006	0.025	0.072	0.156	0.276	0.420	0.569	0.703	0.810	0.887	0.937	0.967	0.984	0.993	0.997	0.999	1.000			
7.4	0.001	0.005	0.022	0.063	0.140	0.253	0.392	0.539	0.676	0.788	0.871	0.926	0.961	0.980	0.991	0.996	0.998	0.999	1.000		
7.6	0.001	0.004	0.019	0.055	0.125	0.231	0.365	0.510	0.648	0.765	0.854	0.915	0.954	0.976	0.989	0.995	0.998	0.999	1.000		
7.8	0.000	0.004	0.016	0.048	0.112	0.210	0.338	0.481	0.620	0.741	0.835	0.902	0.945	0.971	0.986	0.993	0.997	0.999	1.000		
8.0	0.000	0.003	0.014	0.042	0.100	0.191	0.313	0.453	0.593	0.717	0.816	0.888	0.936	0.966	0.983	0.992	0.996	0.998	0.999	1.000	
8.2	0.000	0.003	0.012	0.037	0.089	0.174	0.290	0.425	0.565	0.692	0.796	0.873	0.926	0.960	0.979	0.990	0.995	0.998	0.999	1.000	
8.4	0.000	0.002	0.010	0.032	0.079	0.157	0.267	0.399	0.537	0.666	0.774	0.857	0.915	0.952	0.975	0.987	0.994	0.997	0.999	1.000	
8.6	0.000	0.002	0.009	0.028	0.070	0.142	0.246	0.373	0.509	0.640	0.752	0.840	0.903	0.945	0.970	0.985	0.993	0.997	0.999	0.999	1.000
8.8	0.000	0.001	0.007	0.024	0.062	0.128	0.226	0.348	0.482	0.614	0.729	0.822	0.890	0.936	0.965	0.982	0.991	0.996	0.998	0.999	1.000
9.0	0.000	0.001	0.006	0.021	0.055	0.116	0.207	0.324	0.456	0.587	0.706	0.803	0.876	0.926	0.959	0.978	0.989	0.995	0.998	0.999	1.000
9.5	0.000	0.001	0.004	0.015	0.040	0.089	0.165	0.269	0.392	0.522	0.645	0.752	0.836	0.898	0.940	0.967	0.982	0.991	0.996	0.998	0.999

Siméon Denis Poisson

Born: 21 June 1781 in Pithiviers, France
Died: 25 April 1840 in Sceaux (near Paris), France

Poisson's most important works were a series of papers on definite integrals and his advances in Fourier series.

Originally forced to study medicine, Poisson began to study mathematics in 1798 at the École Polytechnique. His teachers Laplace and Lagrange were to become friends for life. A memoir on finite differences, written when Poisson was 18, attracted the attention of Legendre.

Poisson taught at École Polytechnique from 1802 until 1808 when he became an astronomer at Bureau des Longitudes. In 1809, he was appointed to the chair of pure mathematics in the newly opened Faculté des Sciences.

His most important works were a series of papers on definite integrals and his advances in Fourier series. This work was the foundation of later work in this area by Dirichlet and Riemann.

In Recherchés sur la probabilité des jugements, an important work on probability published in 1837, the Poisson distribution first appeared.

He published between 300 and 400 mathematical works including applications to electricity and magnetism, and astronomy. His Traité de mécanique published in 1811 and again in 1833 was the standard work on mechanics for many years.

His name is attached to a wide area of ideas, for example:- Poisson's integral, Poisson's equation in potential theory, Poisson brackets in differential equations, Poisson's ratio in elasticity, and Poisson's constant in electricity.