

Topic 8 - The Binomial Distribution

Statistics for Managers

June 3, 1999

Bernoulli Trials

The Bernoulli distribution has only two outcomes, Y, success or failure, with values one or zero. The probability of success is p .

Probability Distribution Formula

The probability distribution $f(y)$ is denoted as B (p), the formula is:

$$f(y) = p^y (1-p)^{1-y}$$

Expected Value and Variance

$$E(Y) = p$$

$$VAR(Y) = p(1-p)$$

Example 1. Coin Toss Balanced Coin

Let Y be 0 or 1 depending on the outcome of tail or head, respectively. Since the coin is balanced, p is 0.5.

y	$f(y)$
0	$.5^0 .5^{1-0} = 1 \times \frac{1}{2} = \frac{1}{2}$
1	$.5^1 .5^{1-1} = \frac{1}{2} \times 1 = \frac{1}{2}$

Binomial Distribution

A binomial probability distribution determines the probabilities for the number of wins (successes) Y in n identical wagers (trials) where these conditions are met:

- n identical trials
- outcomes are either a success or a failure
- the probability of success is p

- all trial outcomes are independent events

Probability Distribution Formula

The probability distribution $f(y)$ is denoted as BIN (n, p), the formula is:

$$f(y) = \binom{n}{y} p^y (1-p)^{n-y}$$

Formulas for the Expected Value and Variance

$$E(Y) = np$$

$$VAR(Y) = np(1-p)$$

Example 2. BIN (2, 0.5)

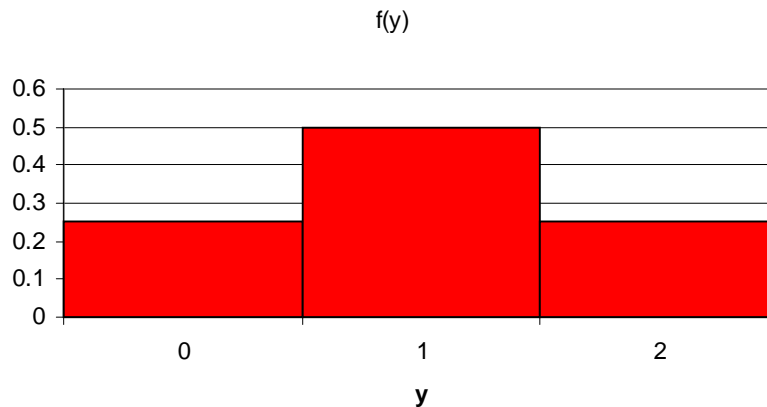
Toss a balanced coin twice, Y is the number of heads.

A carton of parts, half are defective, half are good. Sample two parts replacing them after each draw. Y is the number of defective parts.

Probability Table of BIN (2, 0.5)

y	$f(y)$
2	$\binom{2}{2} 5^2 (1-.5)^{2-2} = 1 \times \frac{1}{4} = \frac{1}{4}$
1	$\binom{2}{1} 5^1 (1-.5)^{2-1} = 2 \times \frac{1}{4} = \frac{1}{2}$
0	$\binom{2}{0} 5^0 (1-.5)^{2-0} = 1 \times \frac{1}{4} = \frac{1}{4}$

Histogram of BIN (2, 0.5)



Example 3. BIN (3, 0.5)

Toss a balanced coin three times, Y is the number of heads.

A carton of parts, half are defective, half are good. Sample three parts replacing them after each draw. Y is the number of defective parts.

A manufacturer makes a particular toy in either red or yellow. After making equally size batches of toys of either color, they mix them for special packaging in sets of three. Collectors value sets of all one color. What proportion of packages have all one color?

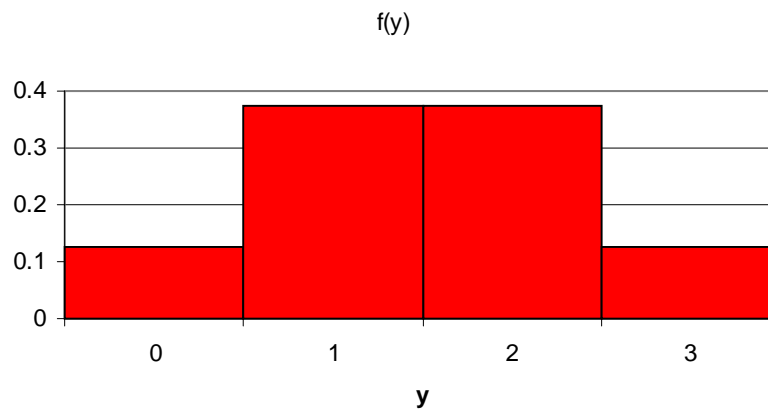
Table BIN (3, 0.5)

y	f(y)
3	$\binom{3}{3} 5^3 (1-.5)^{3-3} = 1 \times \frac{1}{8} = \frac{1}{8}$
2	$\binom{3}{2} 5^2 (1-.5)^{3-2} = 3 \times \frac{1}{8} = \frac{3}{8}$
1	$\binom{3}{1} 5^1 (1-.5)^{3-1} = 3 \times \frac{1}{8} = \frac{3}{8}$
0	$\binom{3}{0} 5^0 (1-.5)^{3-0} = 1 \times \frac{1}{8} = \frac{1}{8}$

Table of equally likely elementary events and BIN (3, 0.5)

y	f(y)	Elementary Events
3	1/8	HHH
2	3/8	HHT HTH THH
1	3/8	HTT THT TTH
0	1/8	TTT

Histogram of BIN (3, 0.5)



Example 4. Defective Auto Parts

Suppose the probability of manufacturing a defective auto part is 10%. What is the probability of shipping at least one defective part in a shipment of 10 parts? In a shipment of 10 parts, Y is the number of defective parts shipped.

Y is BIN (10, 0.1)

$$f(x) = \binom{10}{x} 1^x \cdot 9^{10-x}$$

Table of BIN (10, 0.1)

y	f(y)
0	$\binom{10}{0} 1^0 \cdot 9^{10-0} = 1 \times 1 \times .9^{10} = 0.349$
1	$\binom{10}{1} 1^1 \cdot 9^{10-1} = 10 \times .1 \times .9^9 = 0.387$
2	$\binom{10}{2} 1^2 \cdot 9^{10-2} = 45 \times .1^2 \times .9^8 = 0.194$
3	$\binom{10}{3} 1^3 \cdot 9^{10-3} = 120 \times .1^3 \times .9^7 = 0.0574$
ETC.	

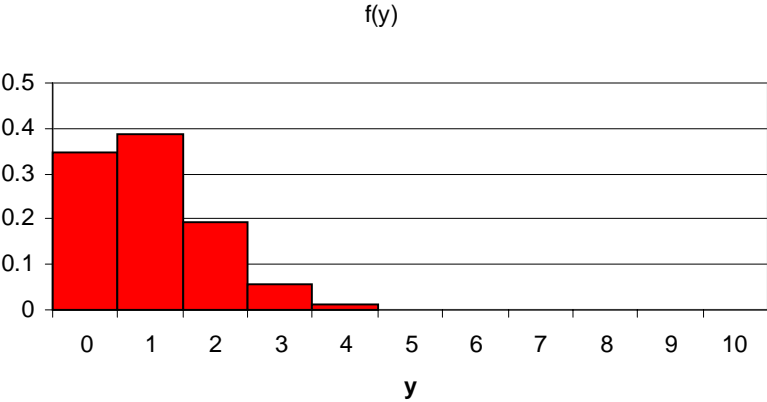
Probability of shipping at least one defective part

The event “shipping zero defective parts” is the complement of “shipping at least one defective part”. Therefore, we may use the basic rule of probability that states for an event A, $P(A') = 1 - P(A)$. Let A = {shipping at least one defective part} and A' = {shipping zero defective parts}. From the above calculation, $P(A') \cong 0.349$. So the probability of shipping at least one defective part is $P(A) = 1 - 0.349 = 0.651$.

Excel worksheet using the built-in function for the binomial distribution

	Excel Formulas	Excel Values
y	f(y)	f(y)
0	=BINOMDIST(0,10,0.1,0)	0.34868
1	=BINOMDIST(1,10,0.1,0)	0.38742
2	=BINOMDIST(2,10,0.1,0)	0.19371
3	=BINOMDIST(3,10,0.1,0)	0.05740
4	=BINOMDIST(4,10,0.1,0)	0.01116
5	=BINOMDIST(5,10,0.1,0)	0.00149
6	=BINOMDIST(6,10,0.1,0)	0.00014
7	=BINOMDIST(7,10,0.1,0)	0.00001
8	=BINOMDIST(8,10,0.1,0)	0.00000
9	=BINOMDIST(9,10,0.1,0)	0.00000
10	=BINOMDIST(10,10,0.1,0)	0.00000

Histogram of BIN (10, 0.1)



Example 5. Marketing claim

The ABC Media Company claims that its advertisements reach 40 percent of the population in its marketing area. After advertising with ABC, the Widget Company did not enjoy increased sales; therefore, it wants to test ABC's 40 per cent claim. Widget Company sampled 10 people and asked them if they had seen the advertisement. Beforehand, Widget decided to reject the 40 percent claim if two or fewer people (20% or less) in the survey saw the advertisement. If ABC's 40 percent claim is true, what is the probability that it is rejected in a given sample?

Y is the number in the survey that responds positively. Y is distributed as BIN (100, 0.40) if the claim is true.

Probability Mass Function and Cumulative Distribution Function for BIN (10, 0.40)

y	$f(y)$	$F(y)$
0	0.00605	0.00605
1	0.04031	0.04636
2	0.12093	0.16729
3	0.21499	0.38228
4	0.25082	0.63310
5	0.20066	0.83376
6	0.11148	0.94524
7	0.04247	0.98771
8	0.01062	0.99833
9	0.00157	0.99990
10	0.00010	1.00000

ABC's claim will be rejected if the event $\{Y \leq 2\}$ occurs. The probability is

$$P(Y \leq 2) = f(0) + f(1) + f(2) = 0.00605 + 0.04031 + 0.12093 = 0.16729.$$

Thus, Widget will be wrongly accusing ABC with probability approximately 16.73 percent, or in other words, about 17 to 83 odds. This represents too high a risk of wrongly accusing someone.

Typically, we like this risk to be a low value somewhere around five to 95 odds, i.e. with probability 0.05. Therefore, what value of y would let $P(Y \leq y) \cong 0.05$?

From the table we see that $P(Y \leq 1) = 0.04636$. This is close enough to 0.05, so the answer is $y = 1$. That is, with probability of 0.05, we take the risk of wrongly confronting ABC when there is one or fewer people who see the advertisement in a sample of 10 people.

Note that the cumulative distribution function was indirectly invoked in this example. Indeed, $P(Y \leq y) = F(y)$. In many instances, the cumulative distribution function is more useful than the probability mass function.

Example 6. Chuck-A-Luck

What is the probability of winning at least one number with the single "one" bet? Use the binomial distribution BIN (3, 1/6). Y is the number "one's". The event $\{Y = 0\}$ indicates there are no wins and is complementary to $\{Y \geq 1\}$, the event of winning at least one number. Therefore,

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - 1 \times (1/6)^0 \times (5/6)^{3-0} = 1 - 125/216 = 91/216.$$

Let Y denote the player's winnings (not net) for a one-dollar bet. What is the expected monetary value?

Expected value using Excel

y	No. "ONE's"	f(y)	yf(y)
0	0	0.57870	0
2	1	0.34722	0.694444444
3	2	0.06944	0.208333333
4	3	0.00463	0.018518519
		E(Y) =	0.921296296

We see that the expected monetary value is approximately 92.13 cents. Therefore, each player loses 7.87 cents to the banker. We found this same monetary value earlier when we calculated the probabilities in a much different and more difficult way.